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# Stochastic Solution of the Linearized Boltzmann Equation

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For the linearized Boltzmann equation with finite cross section, the solution is represented as an integral over the paths of a Markov jump process. The integral is only shown to converge conditionally, where the limiting process is defined by an increasing sequence of stopping times. The notion of local martingale plays an important role. A number of related kinetic models are also mentioned.

**KEY WORDS:** Linearized Boltzmann equation ; jump Markov process ; path integral representation ; multiplicative functional.

## 1. INTRODUCTION

We consider the linearized Boltzmann equation

$$f_t + \xi \cdot f_x = \nu(\xi) \int_R [k(\xi, \eta) f(t, x, \eta) - f(t, x, \xi)] \rho(d\eta) \tag{1}$$

where $\rho(d\xi)$ is a probability measure, $\nu(\xi) > 0$, and $k(\xi, \eta)$ is a kernel of arbitrary sign. This form corresponds to a cutoff hard potential.[1] Our main result yields a representation of $f$:

$$f(t, x, \xi) = \lim_{n \to \infty} E_\xi \left\{ m(t_n) f\left(t - t_n, x - \int_0^{t_n} \xi(s)\, ds, \xi(t_n)\right)\right\} \tag{2}$$

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where $\{\xi(t), t \geqslant 0\}$ is a Markov process determined by $(\nu(\xi), \rho(d\xi))$ and $t_n$ is an increasing sequence of stopping times with $t_n = t$ for $n$ sufficiently large; $m(t)$ is a certain multiplicative functional of $\{\xi(t), t \geqslant 0\}$, determined by the kernel $k$. Formula (2) with $t_n$ replaced by $t$ may not be absolutely convergent. Thus, we are led to the substitute (2). If $E|m(t)| < \infty$ and $f$ is bounded, then (2) reduces to the representation

$$f(t, x, \xi) = E_\xi\left\{m(t)f\left(0, x - \int_0^t \xi(s)\, ds, \xi(t)\right)\right\} \tag{3}$$

This follows from the known theory of multiplicative operator functionals.[2]

The main tool in our proof is a product differentiation formula for piecewise differentiable processes which have simple discontinuities at the jump times of $\xi(t)$. Together with the notion of local martingale,[3] we have the tools to prove (2).

## 2. APPARATUS FOR THE PROOF

We let $R$ denote Euclidean space of any dimension. Let $\rho(d\xi)$ be a probability measure on $R$ and $\nu(\xi)$ a real Borel function which satisfies $0 < \nu(0) \leqslant \nu(\xi) \leqslant k(1 + |\xi|)$.

Let $\{Z_n\}_{n\geqslant 0}$ be independent random variables on a probability triple $(\Omega, B, P)$ with the common law

$$P\{Z_n \in d\xi\} = \rho(d\xi), \qquad n = 0, 1, 2,...$$

Let $\{e_n\}_{n\geqslant 1}$ be random variables conditionally independent of $\{Z_n\}_{n\geqslant 0}$ with the conditional law

$$P\{e_n > t | Z_0, Z_1,...\} = e^{-t\nu(Z_{n-1})}, \qquad t > 0, n = 1, 2,...$$

Define $T_0 = 0$, $T_n = e_1 + \cdots + e_n$, $n \geqslant 1$. Finally, define

$$\xi(t) = Z_{n-1}, \qquad T_{n-1} \leqslant t < T_n, \quad n = 1, 2,...$$

$\{\xi(t), t \geqslant 0\}$ is a conservative regular step Markov process.[5] Let $\mathscr{F}_t = \sigma(\xi(s): s \leqslant t)$.

Now let $k(\xi, \eta)$ be a Borel function on $R \times R$. We define a multiplicative functional by the formula

$$m(t) = \prod_{s \leq t}{}' k(\xi(s^-), \xi(s)) \tag{4}$$

where the product is over all those $s = T_j \leqslant t$. With the same notation for sums we have the following result.

**Lemma 1.** *$m(t)$ satisfies the linear stochastic equation*

$$m(t) = 1 + \sum_{s \leq t}{}' m(s^-)[k(\xi(s^-), \xi(s)) - 1] \tag{5}$$

*Proof.* Clearly $t \to m(t)$ is piecewise constant; but $m(s) - m(s^-) = m(s^-)[k(\xi(s^-), \xi(s)) - 1]$. We sum this equation over the jump times $s \leqslant t$ to arrive at (5). ■

Now let $f(t, x, \xi)$ be a function on $R^+ \times R \times R$ which is differentiable in $(t, x)$. Let

$$Y(t) = f\left(T - t, x - \int_0^t \xi(s)\, ds, \xi(t)\right), \qquad 0 \leqslant t \leqslant T \tag{6}$$

**Lemma 2.** *$Y(t)$ satisfies the stochastic equation*

$$Y(t) = Y(0) - \int_0^t (f_t + \xi \cdot f_x)\left(T - s, x - \int_0^s \xi(u)\, du, \xi(s)\right) ds + \sum_{s \leq t}{}' [Y(s) - Y(s^-)] \tag{7}$$

*Proof.* $t \to Y(t)$ is of class $C^1$ except for $t = T_j$. Thus,

$$Y(t) = Y(0) + \int_0^t \frac{dY}{ds}\, ds + \sum_{s \leq t}{}' [Y(s) - Y(s^-)]$$

But $dY/ds = -(f_t + \xi \cdot f_x)$. The result is now immediate. ■

In order to represent the product $m(t)Y(t)$ in the desired form, we first prove a general formula for product differentiation, which corresponds to a known result (see Théoreme 2, p. 106, Ref. 6). Let $\mathscr{C}$ be the class of all right continuous functions $t \to x(t)$ which are piecewise $C^1$ except for simple discontinuities at the points $t = T_j$. Assume that

$$X(t) = X(0) + \int_0^t F_c(s)\, ds + \sum_{s \leq t}{}' F_d(s)$$

$$Y(t) = Y(0) + \int_0^t G_c(s)\, ds + \sum_{s \leq t}{}' G_d(s)$$

where $F_c$, $G_c$ are integrable on $[0, t]$.

**Lemma 3.** *The product $XY \in \mathscr{C}$ and*

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s)G_c(s)\, ds + \int_d^t Y(s)F_c(s)\, ds + \sum_{s \leq t}{}' \{X(s^-)G_d(s) + Y(s^-)F_d(s) + F_d(s)G_d(s)\}$$

*Proof.* If $t$ is not a discontinuity point, then $XY$ is differentiable and

$$(d/dt)(XY) = X(t)G_c(t) + Y(t)F_c(t)$$

If $t$ is a jump point, then clearly

$$XY(t) - XY(t^-) = X(t^-)G_d(t) + Y(t^-)F_d(t) + F_d(t)G_d(t)$$

But

$$XY(t) - XY(0) = \int_0^t \frac{d}{ds}(XY)\,ds + \sum_{s\le t}{}' [(XY)(s) - (XY)(s^-)]$$

The result clearly follows. ■

We apply this result to the case $X(t) = m(t)$, $Y(t) = f(T - t, x - \int_0^t \xi(s)\,ds, \xi(t))$. After a few manipulations Lemma 3 gives

$$\begin{aligned} X(t)Y(t) &= f(T, x, \xi) - \int_0^t m(s)(f_t + \xi\cdot f_x)\,ds \\ &\quad + \sum_{s\le t}{}' m(s^-)[k(\xi(s^-), \xi(s))f(\xi(s)) - f(\xi(s^-)] \end{aligned} \tag{8}$$

In order to replace this last sum by an integral, we invoke the following lemma, which is a special case of the Lévy system of Watanabe,[7] applied to the Markov process $\{\xi(t), t \geqslant 0\}$.

**Lemma 4.** *If $\varphi(\xi, \eta)$ is a bounded Borel function on $R \times R$, then*

$$E_\xi\left\{\sum_{s\le t}{}' \varphi(\xi(s^-), \xi(s))\right\} = E_\xi \int_0^t \psi(\xi(s))\nu(\xi(s))\,ds \tag{9}$$

*where $\psi(\xi) = \int_R \varphi(\xi, \eta)\rho(d\eta)$.*

*Proof.* The Laplace transform of the left member is

$$\begin{aligned} &\int_0^\infty e^{-\alpha t} E_\xi\left\{\sum_{s\le t}{}' \varphi(\xi(s^-), \xi(s))\right\} dt \\ &= E_\xi \int_0^\infty e^{-\alpha t}\left(\int_0^t \varphi(\xi(s^-), \xi(s))\,dN_s\right) dt \\ &= \frac{1}{\alpha} E_\xi \int_0^\infty e^{-\alpha s}\varphi(\xi(s^-), \xi(s))\,dN_s \\ &= \frac{1}{\alpha} E_\xi\left\{\sum_{n=1}^\infty \varphi(Z_{n-1}, Z_n)e^{-\alpha T_n}\right\} \\ &= \frac{1}{\alpha} E_\xi\left\{\sum_{n=1}^\infty \psi(Z_{n-1})\frac{\nu(Z_{n-1})}{\alpha + \nu(Z_{n-1})} \cdots \frac{\nu(Z_0)}{\alpha + \nu(Z_0)}\right\} \end{aligned}$$

where we have used the conditional distribution of $T_{n+1}$ with respect to

$\{Z_0, Z_1, \ldots\}$. On the other hand, the Laplace transform of the right-hand member is

$$\int_0^\infty e^{-\alpha t} E_\xi\left(\int_0^t \psi(\xi(s))\nu(\xi(s))\,ds\right) dt$$
$$= \frac{1}{\alpha} E_\xi\left\{\int_0^\infty \psi(\xi(s))\nu(\xi(s))e^{-\alpha s}\,ds\right\}$$
$$= \frac{1}{\alpha} E_\xi\left\{\sum_{n=1}^\infty \psi(Z_{n-1})\nu(Z_{n-1})\frac{e^{-\alpha T_n} - e^{-\alpha T_{n-1}}}{\alpha}\right\}$$
$$= \frac{1}{\alpha} E_\xi\left\{\sum_{n=1}^\infty \psi(Z_{n-1})\frac{\nu(Z_{n-1})}{\alpha + \nu(Z_{n-1})} \cdots \frac{\nu(Z_0)}{\alpha + \nu(Z_0)}\right\} \quad \blacksquare$$

In the following, let

$$Z_t^\varphi = \sum_{s \le t}{}' \varphi(\xi(s^-), \xi(s)), \qquad W_t^\varphi = \int_0^t \psi(\xi(s))\nu(\xi(s))\,ds$$

where $\psi(\xi) = \int \varphi(\xi, n)\rho(d\eta)$. Here $Z_t^\varphi$ and $W_t^\varphi$ are both additive functionals of $\{\xi(t), t \geqslant 0\}$ and by Lemma 4, the difference has mean zero. Hence $\{Z_t^\varphi - W_t^\varphi, \mathscr{F}_t\}$ is a martingale if $\varphi$ is bounded.

**Lemma 5.** *Let $\varphi$ be an arbitrary Borel function such that $\psi$ is defined. Then $\{Z_t^\varphi - W_t^\varphi, \mathscr{F}_t\}$ is a local martingale.*

*Proof.* We must show that there exists $\{\tau_m\}$, an increasing sequence of stopping times for $\{\mathscr{F}_t, t \geqslant 0\}$, such that if $A \in \mathscr{F}_s$, then $E_\xi\{Z_{t\wedge\tau_m}^\varphi - W_{t\wedge\tau_m}; A\} = 0$ for $s < t$. In order to prove this, let $\varphi_n(\xi, \eta) = \varphi(\xi, \eta)$ where $|\varphi| \leqslant n$ and zero otherwise. Then $\varphi_n$ is a bounded function and hence $Z_t^{\varphi_n} - W_t^{\varphi_n}$ is a true martingale. If we stop this martingale at $\tau_m \equiv \inf\{t > 0: |\varphi(\xi(t^-), \xi(t))| > m\}$, we have

$$E\{Z_{t\wedge\tau_m}^{\varphi_n} - W_{t\wedge\tau_m}^{\varphi_n}; A\} = 0$$

If we let $n \to \infty$ and use the dominated convergence theorem, the result follows. $\blacksquare$

We are now in a position to state and prove the main result. For notational simplicity, define the Boltzmann generator

$$Lf(x, \xi) = -\xi \cdot f_x + \int_R [k(\xi, \eta)f(x, \eta) - f(x, \xi)]\rho(d\eta)$$

**Theorem.** *Let $f(t, x, \xi)$ be a Borel function which is differentiable in the pair $(t, x)$ with $\int |k(\xi, \eta)f(t, x, \eta)|\rho(d\eta) < \infty$. Then*

$$m(t)f\left(T - t, x - \int_0^t \xi(s)\,ds, \xi(t)\right)$$
$$- \int_0^t m(s)\left[(-f_t + Lf)\left(T - s, x - \int_0^s \xi(u)\,du, \xi(s)\right)\right] ds$$

*is a local martingale.*

*Proof.* By Lemma 3, Equation (8), we have

$$m(t)Y(t) = f(t, x, \xi) - \int_0^t m(s)(f_t + \xi \cdot f_x)\, ds$$

$$+ \sum_{s \le t} m(s^-)[k(\xi(s^-), \xi(s))f(\xi(s)) - f(\xi(s^-))]$$

The last sum is of the form $\sum_{s \le t} m(s^-)\varphi(\xi(s^-), \xi(s))$. By Lemma 5,

$$\sum_{s \le t} \varphi(\xi(s^-), \xi(s)) = \int_0^t \psi(\xi(s))\nu(\xi(s))\, ds + M_t$$

where $\{M_t, \mathscr{F}_t\}$ is a local martingale and $\psi(\xi) = \int [k(\xi, \eta)f(\eta) - f(\xi)]\rho(d\eta)$. Hence

$$\sum_{s \le t} m(s^-)[k(\xi(s^-), \xi(s)f(\xi(s)) - f(\xi(s^-))$$

$$= \int_0^t m(s)\psi(\xi(s))\nu(\xi(s))\, ds + \int_0^t m(s^-)\, dM_s$$

where the last term is again a local martingale. Thus we have

$$m(t)Y(t) = Y(0) + \int_0^t m(s)[-f_t + Lf]\left(T - s, x - \int_0^s \xi(u)\, du, \xi(s)\right) ds$$

$$+ \int_0^t m(s^-)\, dM_s$$

The theorem is now proved.

In order to make connection with the representation (2), let $f(t, x, \xi)$ be a solution of the linearized Boltzmann equation $f_t = Lf$. Then we see that $m(t)f(T - t, x - \int_0^t \xi(s)\, ds, \xi(t))$ is a local martingale. Let $\{\tau_n\}$ be an increasing sequence of stopping times with $\lim_{n \to \infty} \tau_n = \infty$, such that the process stopped at $t \wedge \tau_n$ is a true martingale. Taking expectations gives the representation (2).

## 3. CONCLUDING REMARKS

(a) Let us consider the special case $\nu(\xi) \equiv 1$, $\rho(d\xi) = (2\pi)^{-N/2} \exp(-|\xi|^2/2)$

$$k(\xi, \eta) = 1 + \xi \cdot \eta + (|\xi|^2 - N)(|\eta|^2 - N)/2N$$

Then the right-hand side of (1) is in the form $-I + P$, where $I$ is the identity operator and $P$ is the orthogonal projection onto the subspace of $L^2(R^N, \rho(d\xi))$ spanned by $\{1, \xi, |\xi|^2\}$. This is the *Krook model* of the Boltzmann equation. For this case it is known that[4] $E\{|m(t)|\} < \infty$ and hence (3) holds.

(b) If $\{\nu(\xi), k(\xi, \eta)\}$ corresponds to a cutoff hard potential, it is known[1] that the initial value problem (1) is well posed in $\mathbb{L}^2(R^{2N}, \rho(d\xi)\cdot dx)$. The solution is given by a contraction semigroup in this Hilbert space. But it does not follow from this that the representation (3) is absolutely convergent.

(c) Consider the linearized Boltzmann equation $f_t + \xi\cdot f_x = Qf$ where $Q$ corresponds to a power law potential *without cutoff*[8] (e.g., Maxwellian gas). In this case we have not been able to formulate the stochastic solution. Of course, the solution can be obtained as a limit of (2) when the cutoff parameter converges to zero.

(d) For the nonlinear Boltzmann equation without streaming terms $f_t = B(f,f)$, Tanaka[9] has formulated the stochastic solution in terms of a stochastic integral equation for a discontinuous Markov process. This applies to the case of a Maxwellian gas. But this work does not seem to generalize to the nonlinear Boltzmann equation with the streaming term $\xi\cdot f_x$.

## 4. APPENDIX ON PROBABILISTIC NOTIONS

In this section we give a brief review of the probabilistic notions used in the main body of this paper.

A regular step Markov process on $R^3$ is a stochastic process $\{X(t), t \geqslant 0\}$ constructed from a pair $(\lambda, Q)$ where $\lambda$ is a nonnegative Borel function on $R^3$ and $Q(x, \cdot)$ is a probability measure on the Borel subsets of $R^3$. We define

$$X(t) = Z_n \qquad (\tau_n \leqslant t < \tau_{n+1})$$

where $\{Z_n, \tau_n\}_{n\geqslant 0}$ is a discrete sequence of random variables with the joint law

$$P\{Z_{n+1} \in A | Z_0,\ldots, Z_n\} = Q(Z_n, A) \qquad (n = 0, 1,\ldots)$$

$$P\{\tau_{n+1} - \tau_n > t | \tau_1, Z_1,\ldots, \tau_n, Z_n\} = e^{-t\lambda(Z_n)} \qquad (n = 0, 1,\ldots)$$

with $\tau_0 = 0$, $Z_0 = x \in R^3$. If $\lambda(x) \leqslant$ const, then it can be shown[5] that $\lim_n \tau_n = +\infty$ and thus $X(t)$ is defined for all $t \geqslant 0$. In this case we say that the Markov process is *conservative*.

A *multiplicative functional* of $\{X(t), t \geqslant 0\}$ is a real-valued stochastic process $\{m(t), t \geqslant 0\}$ which satisfies

$$m(t) \in \mathscr{B}\{X(s); s \leqslant t\}$$

$$m(t + s) = m(t)m^+(s) \qquad (m^+(s, w) \equiv m(s, \theta_t w))$$

$$t \to m(t) \text{ is right continuous, a.s.}$$

$$m(0) = 1$$

A wide class of multiplicative functionals can be defined by the formula

$$m(t) = \left[\prod_{\tau_j \leq t} B(X(\tau_{j-1}), X(\tau_j))\right] \exp\left[\int_0^t A(X(s))\, ds\right]$$

where $A$ is a Borel function on $R^3$ and $B$ is a Borel function on $R^3 \times R^3$ which can be supposed to satisfy $B(x, x) = 1$.

A *martingale* is a real-valued stochastic process $\{Y(t), t \geqslant 0\}$ which satisfies

$$Y(t) \in \mathscr{B}\{X(s) : s \leqslant t\}$$

$$E\{Y(t + s) | X(u) : u \leqslant t\} = Y(t)$$

It follows that if $T$ is any stopping time (a nonnegative random variable with $(w: T(w) < t) \in \mathscr{B}\{X(s) : s \leqslant t\})$ then $EY(T) = EY(0)$.

*Examples of martingales*: If $f(x, t)$ is a bounded solution of the integro-differential equation

$$\frac{\partial f}{\partial t}(t, x) = \lambda(x) \int_{R^3} Q(x, dy)[f(y) - f(x)] \tag{A.1}$$

then $f(t_1 - t, X(t))$ is a martingale, $0 \leqslant t \leqslant t_1$.

A *local martingale* is a real-valued stochastic process $\{Y(t), t \geqslant 0\}$ which has the following property: there exists an increasing sequence of stopping times $T_m$ with $\lim_m T_m = +\infty$ such that $Y(t \wedge T_m)$ is a uniformly integrable martingale.

*Example*: If $f(x, t)$ is an unbounded solution of equation (A.1), then $f(t_1 - t, X(t))$ is a local martingale.

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